Natural splines are seldom of any use in cam work. For example, a sixth order natural spline would have the ping and jerk set to zero at the end knots, leaving the beginning and ending velocities and accelerations arbitrary.

Computation

Although cam work using splines seldom requires computing as many unknowns as splines used in other applications, such as graphics, it still can be a daunting task.   For a spline of order m there are a total of $m(k-1)$ linear equations in as many unknowns. One might just directly solve these equations and, for small enough problems, this will work. For large problems a direct approach may give erroneous answers because of roundoff errors or other digital illnesses known to computer scientists. It is recommended that the engineer use some commercially available software, written by computer professionals. Nevertheless, for completeness, we will briefly indicate how the coefficients for the classical spline are computed. Here is a sketch of how one should go about finding the classical sixth order spline. The sketch is brief because this is a book on cams and not on numerical analysis.

The interpolation equations 5.4 and the boundary condition 5.7a immediately give us, for the spline pieces,

$$A_j\left(\theta-\theta_j\right)^5 + B_j\left(\theta-\theta_j\right)^4 + C_j\left(\theta-\theta_j\right)^3 + D_j\left(\theta-\theta_j\right)^2 + E_j\left(\theta-\theta_j\right) + s_j$$

$$\text{for } \theta_j \le \theta \le \theta_{j+1} \text{ and } j = 1,2,\ldots,k-1 \qquad (5.11)$$

Believe it or not, we are now going to add more unknowns, so that we have symmetry in our equation set. We write equations 5.8a, b, and c to be

$$A_{k-1}h_{k-1}^5 + B_{k-1}h_{k-1}^4 + C_{k-1}h_{k-1}^3 + D_{k-1}h_{k-1}^2 + E_{k-1}h_{k-1} + s_{k-1} = s_k$$

$$5A_{k-1}h_{k-1}^4 + 4B_{k-1}h_{k-1}^3 + 3C_{k-1}h_{k-1}^2 + 2D_{k-1}h_{k-1} + E_{k-1} = E_k \qquad (5.12)$$

$$10A_{k-1}h_{k-1}^3 + 6B_{k-1}h_{k-1}^2 + 3C_{k-1}h_{k-1} + D_{k-1} = D_k$$

so that now the boundary conditions for b are:

$$F_k = s(b)$$
$$E_k = s'(b) \qquad (5.13)$$
$$D_k = s''(b)$$

Equations 5.6a, -b, and -c can be solved to obtain three coefficients in terms of the other two. Thanks to the symmetry, the solution also applies to equation 5.12. The solution is:

$$A_j = \frac{1}{h_j^4}\left(-h_j D_j + h_j D_{j+1} - 3E_j - 3E_{j+1} + 6\frac{F_{j+1}-F_j}{h_j}\right) \quad \text{for } j = 1, 2, \ldots, k$$

$$B_j = \frac{1}{h_j^3}\left(3h_j D_j - 2h_j D_{j+1} + 8E_j + 7E_{j+1} - 15\frac{F_{j+1}-F_j}{h_j}\right) \quad \text{for } j = 1, 2, \ldots, k \qquad (5.14)$$

$$C_j = \frac{1}{h_j^2}\left(-3h_j D_j + h_j D_{j+1} - 6E_j - 4E_{j+1} + 10\frac{F_{j+1}-F_j}{h_j}\right) \quad \text{for } j = 1, 2, \ldots, k$$

Substituting these equations for both the left and right side of equations 5.6d and -e, we get the set of equations

$$-\frac{1}{2h_{j-1}}D_{j-1}+\frac{3}{2}\left(\frac{1}{h_{j-1}}+\frac{1}{h_j}\right)D_j-\frac{1}{2h_j}D_{j+1}$$

$$-\frac{2}{h_{j-1}^2}E_{j-1}-3\left(\frac{1}{h_{j-i}^2}-\frac{1}{h_j^2}\right)E_j+\frac{2}{h_j^2}E_{j+1} \qquad (5.15)$$

$$=5\left[\left(\frac{F_{j-1}}{h_{j-1}^3}+\frac{F_{j+1}}{h_j^3}\right)-\left(\frac{F_j}{h_{j-1}^3}+\frac{1}{h_j^3}\right)\right], \qquad \text{for } j=1,2,\dots,k$$

$$\frac{2}{h_{j-i}^2}D_{j-1}-3\left(\frac{1}{h_{j-i}^2}+\frac{1}{h_j^2}\right)D_j-\frac{2}{h_j^2}D_{j+1}$$

$$+\frac{7}{h_{j-i}^3}E_{j-1}+8\left(\frac{1}{h_{j-i}^3}-\frac{1}{h_j^3}\right)E_j+\frac{7}{h_j^3}E_{j+1} \qquad (5.16)$$

$$=15\left(\frac{F_{j+1}}{h_j^4}-\frac{F_{j-1}}{h_{j-1}^4}\right), \qquad \text{for } j=1,2,\dots,k$$

and the boundary equations 5.7b, 5.7c, and 5.13). After solving this set of linear equations, we evaluate equations 5.14 to get the remaining coefficients. We must point out that equations 5.14, 5.15, and 5.16 are for the sixth order classical spline only. Other order splines will use other equations. We will now go over everything that was said above using a specific example.

EXAMPLE 5-1

Using a Classical Spline for a Single-Dwell Cam.

Problem: Consider the following single-dwell cam specification:

rise	1 inch (25.4 mm) in 90°
fall	1 inch (25.4 mm) in 90°
dwell	at zero displacement for 180° (low dwell)
cam ω	15 rad/sec

Solution:

1   The first step in constructing a classical spline is to pick knot locations. The first row of Table 5-2 shows our knot selections. We place knots at 0, 45, 90, 135, and 180°. The second row of the table shows the displacements we want at each of the knots. The knots

TABLE 5-2   Knot Locations and Boundary Conditions for 1st Try, Example 5-1

Function	Knot Locations (deg)				
	0	45	90	135	180
Displacement	0	0.5	1	0.5	0
Velocity	0	–	–	–	0
Acceleration	0	–	–	–	0

5

at 45, 90 and 135° are interior knots. The knots at 0 and 180° are end knots. The knots at 0, 90, and 180° are demanded by the problem in that we are asked to design a cam motion that has a displacement of 0 at 0°, 1 inch at 90° and returns to 0 at 180°. The interior knots at 45 and 135° are not necessary. We have arbitrarily selected them and could have selected more knots or fewer knots or even no knots other than those demanded by the problem. The reason that we added the two extra knots was to give us control over the velocity, acceleration and jerk, as will soon be shown. We might also mention that we chose our knots to be equally spaced from each other. This will simplify our equations. Such a spline is called a **uniform spline**.

2   The next step in our classical spline construction is to choose the order of the spline. We choose a classical spline of sixth order to solve our problem. We therefore expect to have a very smooth curve with continuous derivatives up beyond jerk, except at the end knots. At the end knots we can match only the displacement, velocity, and acceleration of the dwell. There will be a jump discontinuity at the end knots in the jerk, but this is acceptable.

3   For computational reasons, we will use radians rather than degrees. The polynomial pieces are:

$$p_1(\theta) = A_1\theta^5 + B_1\theta^4 + C_1\theta^3 + D_1\theta^2 + E_1\theta + F_1$$
$$p_2(\theta) = A_2(\theta - \pi/4)^5 + B_2(\theta - \pi/4)^4 + C_2(\theta - \pi/4)^3 + D_2(\theta - \pi/4)^2 + E_2(\theta - \pi/4) + F_2$$
$$p_3(\theta) = A_3(\theta - \pi/2)^5 + B_3(\theta - \pi/2)^4 + C_3(\theta - \pi/2)^3 + D_3(\theta - \pi/2)^2 + E_3(\theta - \pi/2) + F_3$$
$$p_4(\theta) = A_4(\theta - 3\pi/4)^5 + B_4(\theta - 3\pi/4)^4 + C_4(\theta - 3\pi/4)^3 + D_4(\theta - 3\pi/4)^2 + E_4(\theta - 3\pi/4) + F_4$$

(a)

with a total of 24 unknown coefficients for the 4 polynomial pieces.

4   The interpolation equations are:

$$F_2 = 0.5 \qquad\qquad F_3 = 1.0 \qquad\qquad F_4 = 0.5 \qquad\qquad \text{(b)}$$

and equations 5.14 become:

$$h_1 = h_2 = h_3 = h_4 = \pi/4, \quad d_1 = 2/\pi, \ d_2 = 4/\pi, \quad d_3 = 2/\pi, \ d_4 = 0$$

$$A_j = \frac{256}{\pi^4}\left[-\frac{\pi}{4}D_j + \frac{\pi}{4}D_{j+1} - 3E_j - 3E_{j+1} + \frac{24}{\pi}\left(F_{j+1} - F_j\right)\right], j = 1, 2, 3, 4$$

$$B_j = \frac{128}{\pi^3}\left[\frac{3\pi}{4}D_j - \frac{\pi}{2}D_{j+1} + 8E_j + 7E_{j+1} - \frac{60}{\pi}\left(F_{j+1} - F_j\right)\right], j = 1, 2, 3, 4 \qquad (c)$$

$$C_j = \frac{16}{\pi^2}\left[-\frac{3\pi}{4}D_j + \frac{\pi}{4}D_{j+1} - 6E_j - 4E_{j+1} + \frac{40}{\pi}\left(F_{j+1} - F_j\right)\right], j = 1, 2, 3, 4$$

Equations 5.15 and 5.16 become:

$$2D_1 - 12D_2 + 2D_3 + \frac{32}{\pi}E_1 - \frac{32}{\pi}E_3 = 0$$

$$2D_2 - 12D_3 + 2D_4 + \frac{32}{\pi}E_2 - \frac{32}{\pi}E_4 = \frac{320}{\pi^2}$$

$$2D_3 - 12D_4 + 2D_5 + \frac{32}{\pi}E_3 - \frac{32}{\pi}E_5 = 0 \qquad (d)$$

$$D_1 - D_3 + \frac{14}{\pi}E_1 + \frac{32}{\pi}E_2 + \frac{14}{\pi}E_3 = \frac{120}{\pi^2}$$

$$D_2 - D_4 + \frac{14}{\pi}E_2 + \frac{32}{\pi}E_3 + \frac{14}{\pi}E_4 = 0$$

$$D_3 - D_5 + \frac{14}{\pi}E_3 + \frac{32}{\pi}E_4 + \frac{14}{\pi}E_5 = -\frac{120}{\pi^2}$$

The boundary equations are:

$F_1 = 0$	$E_1 = 0$	$D_1 = 0$	at the start of the rise, $\theta = 0$	(e)
$F_5 = 0$	$E_5 = 0,$	$D_5 = 0$	at the end of the rise, $\theta = \pi$	(f)

The final spline is:

$$s(\theta) = \begin{cases} 1.0009\theta^5 - 3.2264\theta^4 + 2.9487\theta^3, \quad 0 \le \theta \le \theta/4 \\[4pt] -0.1942(\theta - \pi/4)^5 + 0.7040(\theta - \pi/4)^4 - 1.0136(\theta - \pi/4)^3 \\ \qquad\qquad - 0.14474(\theta - \pi/4)^2 + 1.1084(\theta - \pi/4) + 0.5, \quad \pi/4 \le \theta \le \pi/2 \\[4pt] 0.1942(\theta - \pi/2)^5 - 0.0587(\theta - \pi/2)^4 - 0.8685(\theta - \pi/2)^2 + 1.0, \quad \pi/2 \le \theta \le 3\pi/4 \\[4pt] -1.0009(\theta - 3\pi/4)^5 + 0.7040(\theta - 3\pi/4)^4 + 1.0136(\theta - 3\pi/4)^3 \\ \qquad\qquad - 0.1447(\theta - 3\pi/4)^2 - 1.1084(\theta - 3\pi/4) + 0.5, \quad 3\pi/4 \le \theta \le \pi \\[4pt] 0, \quad \text{elsewhere} \end{cases} \qquad (g)$$